project tips:

① Clarity: define your problem setup mathematically
(all symbols need to be well-defined)

* unifying notations from multiple papers

② Technical Quality:

* identify & discuss the key learning theory qns your project tries to answer.
* lit survey: avoid "laundry list". discuss relationships b/w papers.
* implementation: for plots. make sure have error bars.

③ Feel free to follow up by email.

Online newton step:

- $\Omega$. $\max_{u,v \in \Omega} \|u-v\|_2 \leq B$.

$\{f_t\}_{t=1}^{T}$ are all $\alpha$-exp-concave & L-Lip.

$\lambda$: parameter for the algorithm.

- initialize $w_1$ to be any pt in $\Omega$:

- For $t = 1, 2 \dots T$:
  - show $w_t$.
  - receive $f_t$.
  - update $w_t$: $g_t \in \partial f_t(w_t)$

$$w_{t+1} = \underset{w \in \Omega}{\operatorname{argmin}} \ \langle w, \underline{g_t} \rangle + D_{\underline{\varphi_t}}(w, w_t).$$

$$\varphi_t(w) = \frac{1}{2} \|w\|_{A_t}^2, \quad A_t = \lambda I + \tilde{\alpha} \cdot \sum_{s=1}^{t} g_s g_s^T$$

( Alternatively:

$$w'_{t+1} = w_t - A_t^{-1} \cdot g_t$$

$$w_{t+1} = \underset{w \in \Omega}{\operatorname{argmin}} \ \| w - w'_{t+1} \|_{A_t}. \ )$$

Analysis of online newton step:

<u>Thm</u>: $\lambda = \frac{1}{\tilde{\alpha} B^2}$. $\tilde{\alpha} = \min\left(\frac{1}{8 \cdot B \cdot L}, \frac{\alpha}{2}\right)$.

online newton step gives regret

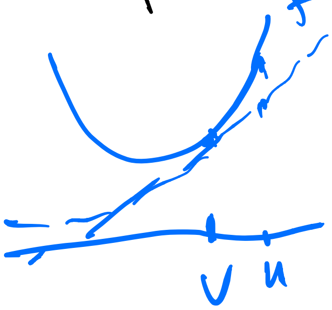
$$R_T(\Omega) \le O\left(\frac{1}{\tilde{\alpha}} d \ln T\right)$$

$$= O\left(\left(\frac{1}{\alpha} + L\cdot B\right)\cdot d \ln T\right).$$

<u>Key Lemma</u>: If $f$ is $\alpha$-exp-concave & $L$-Lip. then, for any $u, v \in \Omega$,

can be replaced with $g_v \in \partial f(v)$.

$$f(u) \geq f(v) + \langle \nabla f(v), u - v\rangle + \frac{\tilde{\alpha}}{2} \cdot (u-v)^T \nabla f(v) \nabla f(v)^T (u-v).$$




Pf of thm:

similar to "linearization" in OMD pf:

$A_t - A_{t-1}$

$$f_t(w_t) - f_t(u) \leq \langle \nabla f_t(w_t), w_t - u\rangle - \frac{\tilde{\alpha}}{2}(w_t-u)^T g_t g_t^T (w_t-u)$$

Additional useful properties of exp concave fns:

① . ~~f is $\alpha$-exp-concave~~ $\beta \le \alpha$

$\forall w.\ \nabla^2 f(w) \succcurlyeq \alpha \cdot \nabla f(w) \nabla f(w)^T$

$\Rightarrow$ f is $\beta$-exp-concave ?

$$\exp(-\alpha f(w)) \longrightarrow \exp(-\beta f(w)).$$

$$\underset{g(w)}{\parallel} \qquad\qquad \underset{(g(w))^{\frac{\beta}{\alpha}}.}{\parallel}$$

② f is twice differentiable & $\underline{\lambda\text{-sc.}}$ L-lip

$\|\cdot\|_2$.

$$\forall w.\ \nabla^2 f(w) \succcurlyeq \lambda \cdot I.$$

key observation: for unit vector $a$, $I \succcurlyeq aa^T$

<u>pf</u>: expand $a$ to an orthonormal basis.

$\overset{=a}{v_1}, v_2 \cdots v_d$

$$I = (v_1 \cdots v_d)\begin{pmatrix} v_1^T \\ \vdots \\ v_d^T \end{pmatrix} = \sum_{i=1}^d v_i v_i^T$$

$$I - aa^T = \sum_{i=2}^d v_i v_i^T \succcurlyeq 0.$$

$$\sum^d (x^T v_i)^2 \geq 0$$

$\forall x \quad x^T\left(\sum_{i=1}^{d} u_i v_i^T\right) x = \sum_{i=1} (\ldots$

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$f$ is $\frac{\lambda}{L^2}$-exp-concave.

continuing the ONS analysis:

$$f_t(w_t) - f_t(u) \le \underbrace{\langle g_t, w_t - u\rangle}_{①} - \frac{1}{2}\|w_t - u\|^2_{A_t - A_{t-1}}$$

OMD: $\quad w_{t+1} = \underset{w\in\Omega}{\mathrm{argmin}} \langle w, g_t\rangle + D_\psi(w, w_t)$

$$\left.\begin{aligned}\langle g_t, w_t - u\rangle &\le D_\psi(u, w_t) - D_\psi(u, w_{t+1}) \\ &\quad + \frac{1}{2}\|g_t\|^2_* .\end{aligned}\right\} (*)$$

$\psi(w) = \frac{1}{2}\|w\|^2_{A_t}$. $\quad \|\cdot\| = \|\cdot\|_{A_t} \quad \|\cdot\|_* = \|\cdot\|_{A_t^{-1}}$

$(*)$

$$\Rightarrow ① \le \frac{1}{2}\|u - w_t\|^2_{A_t} - \frac{1}{2}\|u - w_{t+1}\|^2_{A_t} + \frac{1}{2}\|g_t\|^2_{A_t^{-1}}$$

$\Rightarrow f_t(w_t) - f_t(u)$

$D_t \qquad D_{t+1}$

$x$.

$\|x\|^2_A - \|x\|^2_B$

$= \|x\|^2_{A-B}$

$$\leq \frac{1}{2}\|u-w_t\|^2_{A_{t-1}} - \frac{1}{2}\|u-w_{t+1}\|^2_{A_t} \quad x^T(A-B)x.$$

$$+\frac{1}{2}\|g_t\|^2_{A_t^{-1}}. \qquad D_1 - D_{T+1}.$$

summing over $t$'s.

$$R_T(u) \leq \underbrace{\frac{\lambda}{2}\|u-w_1\|_2^2}_{①} + \underbrace{\frac{1}{2}\sum_{t=1}^{T}\|g_t\|^2_{A_t^{-1}}}_{②} \quad \tilde{\alpha}^{-1}B_t^{-1}$$

$$A_t = \tilde{\alpha} B_t$$

bounding ① : $\leq \frac{\lambda}{2} B^2$.

$$= \frac{1}{2}\cdot\frac{1}{\tilde{\alpha}B^2}\cdot B^2$$

$$= \frac{1}{2\tilde{\alpha}}.$$

bounding ② : $B_t = \frac{1}{\tilde{\alpha}} A_t$

$$= \frac{\lambda}{\tilde{\alpha}} I + \sum_{s=1}^{t} g_s g_s^T$$

$$② = \frac{1}{2\tilde{\alpha}} \sum_{t=1}^{T} \|g_t\|^2_{B_t^{-1}}$$

$\mu = \frac{\lambda}{\tilde{\alpha}}$.

$D = L$.

The Elliptic potential lemma:

suppose $u_1 \cdots u_T \in \mathbb{R}^d$.

$V_t = \mu I + \sum_{s=1}^{t} u_s u_s^T$ then

$$\sum_{t=1}^{T} \|u_t\|^2_{V_t^{-1}} \overset{①}{\leq} \ln \frac{\det(V_T)}{\det(V_0)} \quad \mu I.$$

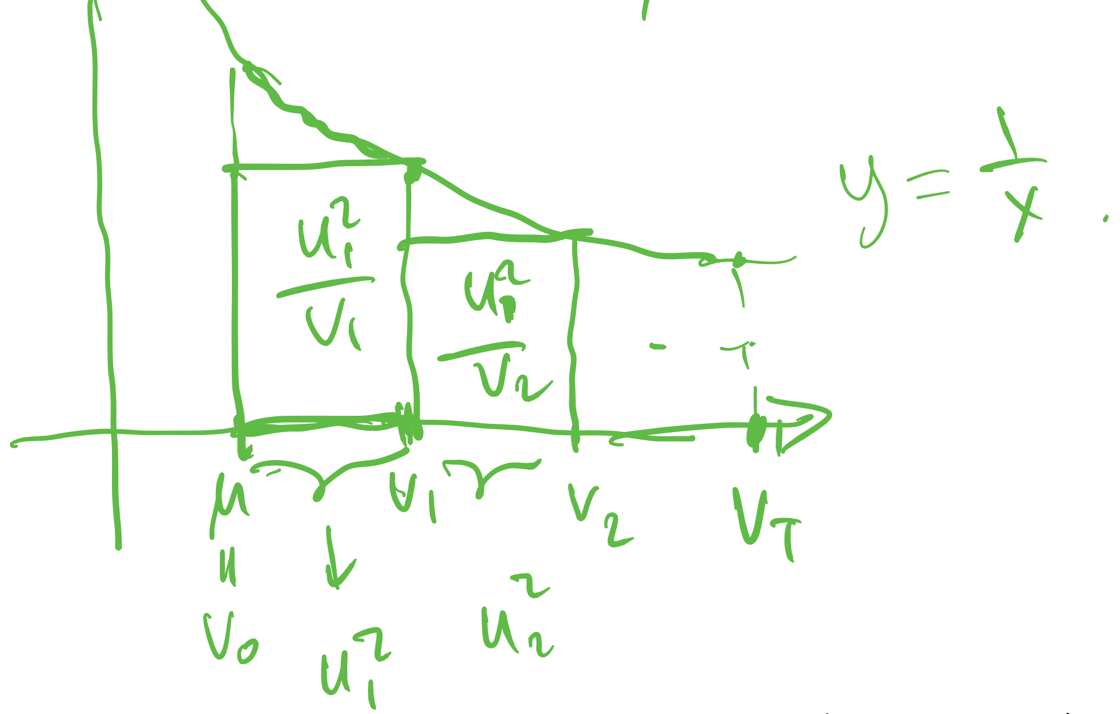
If all $\|u_t\| \leq D$.

$$\overset{②}{\leq} d \ln\left(1 + \frac{D^2 T}{d\mu}\right).$$

Ex: 1-d. $V_t = \mu + \sum_{s=1}^{t} u_s^2$.

$$\sum_{t=1}^{T} \frac{u_t^2}{V_t} \leq \int_{\mu}^{V_T} \frac{1}{x} dx \quad \Theta(\ln T)$$

$$= \ln \frac{V_T}{\mu}.$$




plugging the elliptic pot. lemma to ②:

$$② \le \frac{d}{2\tilde{\alpha}} \ln\left(1 + \frac{L^2 T \tilde{\alpha}}{d \lambda}\right)$$

$$\frac{1}{\tilde{\alpha} B^2}$$

$$= \frac{d}{2\tilde{\alpha}} \ln\left(1 + \frac{L^2 T \tilde{\alpha}^2 B^2}{d}\right)$$

$$\le \frac{1}{64 B^2 L^2}$$

$$\le O\left(\frac{d}{\tilde{\alpha}} \ln T\right).$$

In conclusion $R_T(u) \le ① + ②$

$$\le O\left(\frac{d}{\lambda} \ln T\right)$$ $\boxtimes$

proof of Elliptic potential lemma:

show that $\forall t$:

$$\|u_t\|^2_{V_t^{-1}} \le \ln \frac{\det(V_t)}{\det(V_{t-1})} \quad (*)$$

$$\left(1\text{-}d:\ \frac{u_t^2}{v_t} \le \int_{v_{t-1}}^{v_t} \frac{1}{x}\, dx\right).$$

pf of $(*)$:

$\det(A^{-1}) = \frac{1}{\det(A)}$

$$\text{RHS} = -\ln \frac{\det(V_{t-1})}{\det(V_t)}$$

$\det(AB) = \det(A)\det(B)$

$$= -\ln \det(V_{t-1}) \cdot \det(V_t^{-1})$$

$$= -\ln \det(\underline{V_t^{-1}} \underbrace{V_{t-1}}_{I})$$

$$V_{t-1} = V_t - u_t u_t^T$$

$$= -\ln \det(\underbrace{I}_{V_t} - \underline{V_t^{-1}} u_t u_t^T)$$

$$V_t^{-1} = V_t^{-\frac{1}{2}} \cdot V_t^{-\frac{1}{2}}$$

$$= -\ln \det\left( V_t^{-\frac{1}{2}} ( \underline{V_t^{+\frac{1}{2}}} - V_t^{-\frac{1}{2}} u_t u_t^T ) \right)$$

$$= -\ln \det\left( V_t^{-\frac{1}{2}} ( I - V_t^{-\frac{1}{2}} u_t u_t^T V_t^{-\frac{1}{2}} ) V_t^{\frac{1}{2}} \right)$$

$$= -\ln \det( I - \underbrace{V_t^{-\frac{1}{2}} u_t}_{a} \underbrace{u_t^T V_t^{-\frac{1}{2}}}_{a^T} )$$

Fact: $\underbrace{\det(I - aa^T)}$ . $\quad u_t^T V_t^{-\frac{1}{2}} \cdot V_t^{-\frac{1}{2}} u_t$

$\phantom{xx}\cancel{\quad} \; V_t^{-1}$

$= 1 - a^T a$ .

( idea: $\hat{a} = \frac{a}{\|a\|}$ .

$$I = \sum_{i=1}^{d} v_i v_i^T \qquad v_1 = \hat{a}$$

$$I - aa^T = (1 - a^T a)\, v_1 v_1^T + \sum_{i=2}^{d} v_i v_i^T$$

eigenvalues $= \left( (1 - a^T a),\ \underbrace{1 \cdots 1}_{d-1} \right)$

Fact

$$\Rightarrow \; = -\ln \underbrace{\left(1 - \|u_t\|^2_{V_t^{-1}}\right)}_{\leq -\|u_t\|^2_{V_t^{-1}}}$$

Fact: $\ln(1+x) \leq x$ .

Fact

$$\Rightarrow \; \geq \|u_t\|^2_{V_t^{-1}} .$$

Pf of ineq ②:

$$V_T = \mu I + \sum_{t=1}^{T} u_t u_t^T$$

$$\ln \frac{\det(V_T)}{\det(\mu I)}$$

$$= \ln \det \left( I + \frac{1}{\mu} \sum_{t=1}^{T} u_t u_t^T \right).$$

Fact: $M$: $\lambda_1 \cdots \lambda_d$.

$$\sum_{i=1}^{d} \lambda_i = \mathrm{tr}(M) = \sum_{i=1}^{d} M_{ii}.$$

$$\det(M) = \prod_{i=1}^{d} \lambda_i.$$

$$\overset{\text{AM-GM}}{\leq} \left( \frac{\sum_i \lambda_i}{d} \right)^d$$

for all $i$

$$\sum_i \sum_t u_{t,i}^2 \leq T D^2$$

$$\|u_t\|_2 \leq D$$

$$= \left(\frac{\mathrm{tr}(M)}{d}\right)^d .$$

Fact $\Rightarrow$

$$\le d \ln \left( \frac{\mathrm{tr}\left(I + \frac{1}{\mu} \sum_{t=1}^{T} u_t u_t^T\right)}{d} \right)$$

$$\le d \ln \left( 1 + \frac{T D^2}{d \mu} \right) . \quad \boxtimes$$

Pf of quadratic lower lemma: see Piazza for link.

Online learning w/ bandit feedback:

§: stochastic multi-armed bandits.

Recall product recommendation:

For $t = 1, 2, \ldots T$: $\quad \ell_t(i)$:

environment draws $\ell_t \in [0,1]^K$ (not revealed to learner)

(arm)

learner selects action $a_t \in \{1 - K\}$.

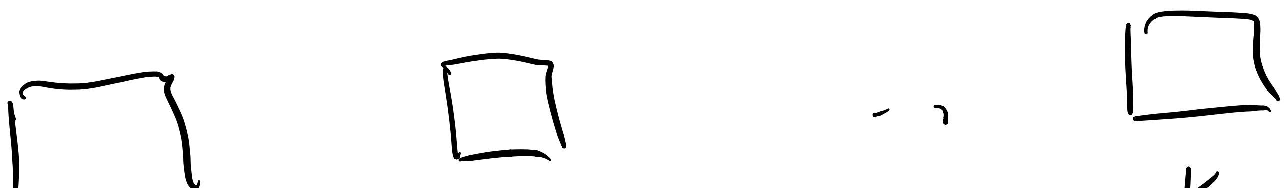
learner suffers & sees loss $\ell_t(a_t)$.

and it does not see $\{\ell_t(a) : a \neq a_t\}$.

Assume: $\forall a$: $(\ell_t(a))_{t=1}^{T}$ are drawn iid from a distn over $[0,1]$ with mean $\ell(a)$.

Goal: minimize: $\sum_{t=1}^{T} \ell_t(a_t)$.

Ex: slot machine game



1 2 ... K.

challenge:

① learn which machine is rewarding → explore

② play machines that are rewarding → exploitation