CSC 480/580 Homework 0

Instructions:

- This is a background self-test on the type of math we will encounter in class. If you find lots of questions intimidating, you will have to spend nontrivial extra efforts to catch up, or you can take this course next time after catching up the backgrounds.
- Submit your homework on time (due: **1/18 5pm**) to Gradescope. NO LATE DAYS, NO LATE SUBMISSIONS ACCEPTED.
- The submission must be one single PDF file (use Acrobat Pro from the UA software library if you need to merge multiple PDFs).
  - You can use word processing software like Microsoft Word or LaTeX.
  - You can also hand-write your answers and then scan it. If you use your phone camera, I recommend using TurboScan (smartphone app) or similar ones to avoid looking slanted or showing the background.
  - Watch the video and follow the instruction: https://youtu.be/KMPoby5g_nE .
- I require that you spend some time to figure out an answer to the homework. If you failed to figure out the answer, please explain what you have done to find an answer and where you get stuck.
  - Don't: "I googled it but nothing came up."
  - Do: "I read material A, and there is this statement B that seems to help, but when I tried to apply it, C became an issue due to independence. I tried to apply D that does not require the independence, but there was an issue of E."
- This homework will not be part of the homework score. However, The participation score will be deducted (-2pts out of 10) if there are no nontrivial efforts to solve it.

1 Vectors and Matrices

Consider the matrix X and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$X = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

1. Compute $\mathbf{y}^\top X \mathbf{z}$.

2. Is X invertible? If so, give the inverse, and if no, explain why not.

2 Calculus

1. If $y = e^{x^2} + \tan(z)x^{6z} - \ln(\frac{8x+16}{x^4})$, what is the partial derivative of y with respect to x?

3 Probability and Statistics

Consider a sequence of data $S = (0, 1, 1, 0, 0, 1, 1)$ created by independently flipping a (possibly-biased) coin X seven times, where $X = 0$ denotes that the coin turned up heads and $X = 1$ denotes that it turned up tails. Denote by $p := P(X = 1)$ the *bias* of the coin.

1. What is the probability of observing this data, assuming it was generated by flipping a biased coin with $p = 0.5$?
2. Note that the probability of this data sample could be greater if the value of p was not 0.5, but instead some other value. What is the value of p that maximizes the probability of observing S? Please justify your answer.
3. Consider the following joint probability table where both A and B are binary random variables:

a	b	$P(A = a, B = b)$
0	0	0.4
0	1	0.3
1	0	0.2
1	1	0.1

   (a) What is $P(A = 1 \vee B = 1)$?

   (b) What is $P(A = 1|B = 0)$?

4 Big-O Notation

For each pair (f, g) of functions below, answer both of the following questions: a) Is $f(n) = O(g(n))$?, b) Is $g(n) = O(f(n))$? Justify your answers.

1. $f(n) = \frac{n}{2}$, $g(n) = \log_2(n)$.
2. $f(n) = \ln(n)$, $g(n) = \log_2(n)$.
3. $f(n) = n^{100}$, $g(n) = 100^n$.

5 Probability and Random Variables

5.1 Probability

For each of the statement below, state whether it is true or false; if false, provide a counterexample. Here Ω denotes the sample space and A^C denotes the complement of the event A. Hint: for counterexamples, you may consider $\Omega = \{1, 2, 3, 4\}$ and assign suitable probability mass to each element in Ω.

1. For any $A, B \subseteq \Omega$, $P(A|B)P(B) = P(B|A)P(A)$.

2. For any $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B) - P(A|B)$.

3. For any $A, B, C \subseteq \Omega$ such that $P(B \cup C) > 0$, $\frac{P(A \cup B \cup C)}{P(B \cup C)} \geq P(A|B \cup C)P(B \cup C)$.

4. For any $A, B \subseteq \Omega$ such that $P(B) > 0, P(A^c) > 0$, $P(B|A^C) + P(B|A) = 1$.

5.2 Discrete and Continuous Distributions

Match the distribution name to its probability density / mass function. Below, $\boldsymbol{x} \in \mathbb{R}^k$.

(a) Laplace
(b) Multinomial
(c) Poisson
(d) Dirichlet
(e) Gamma

(f) $f(\boldsymbol{x}; \boldsymbol{\Sigma}, \boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right)$
(g) $f(x; n, \alpha) = \binom{n}{x}\alpha^x(1-\alpha)^{n-x}$ for $x \in \{0, \ldots, n\}$; 0 otherwise
(h) $f(x; b, \mu) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$
(i) $f(\boldsymbol{x}; n, \boldsymbol{\alpha}) = \frac{n!}{\Pi_{i=1}^k x_i!}\Pi_{i=1}^k \alpha_i^{x_i}$ for x_i's $\in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_i = n$; 0 otherwise
(j) $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}$ for $x \in (0, +\infty)$; 0 otherwise
(k) $f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$ for x_i's $\in (0, 1)$ and $\sum_{i=1}^k x_i = 1$; 0 otherwise
(l) $f(x; \lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$ for all $x \in \mathbb{Z}^+$; 0 otherwise

5.3 Mean and Variance

1. Consider a random variable which follows a Binomial distribution: $X \sim \text{Binomial}(n, p)$.
   (a) What is the mean of the random variable?

   (b) What is the variance of the random variable?

2. Let X be a random variable and $\mathbb{E}[X] = 2, \text{Var}(X) = 2$. Compute the following values:
   (a) $\mathbb{E}[3X]$
   (b) $\text{Var}(3X)$
   (c) $\text{Var}(X + 3)$

5.4 Mutual and Conditional Independence

For the questions below, you may assume that the outcome space Ω is finite.

1. If X and Y are independent random variables, show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
2. If X and Y are independent random variables, show that $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.
   Hint: you can use the fact that $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y)$
3. (a) If we roll two dice independently of each other, will the outcome of the first die tell us something new about the outcome of the second die?

   (b) Suppose I rolled two dice independently, and I tell you that the sum of the outcomes of the two dice are an even number (but I do not tell you the outcomes of the two dice). Given this information, is the outcome of the second die independent of the outcome of the first die? Prove or disprove.

6 Linear Algebra

6.1 Norms

Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$ that satisfy the following:

1. $||\mathbf{x}||_1 \leq 1$ (Recall that $||\mathbf{x}||_1 = \sum_i |x_i|$)
2. $||\mathbf{x}||_2 \leq 1$ (Recall that $||\mathbf{x}||_2 = \sqrt{\sum_i x_i^2}$)
3. $||\mathbf{x}||_\infty \leq 1$ (Recall that $||\mathbf{x}||_\infty = \max_i |x_i|$)

6.2 Geometry

Prove: The smallest Euclidean distance from the origin to some point $\mathbf{x}$ in the hyperplane $\mathbf{w}^\top \mathbf{x} + b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$. You may assume $\mathbf{w} \neq 0$.

7 Programming Skills

Sampling from a distribution. For each question, submit a scatter plot (you will have 3 plots in total). Make sure the axes for all plots have the same ranges. Email your code to csc580homeworks@gmail.com.

1. Draw 100 samples $\mathbf{x} = [x_1, x_2]^\top$ from a 2-dimensional Gaussian distribution $\mathcal{N}(\mu, \Sigma)$ with mean $\mu = (0, 0)^T$ and identity covariance matrix $\Sigma = I$, i.e., $p(\mathbf{x}) = \frac{1}{2\pi} \exp\left(-\frac{\|\mathbf{x}\|_2^2}{2}\right)$, and make a scatter plot (x_1 vs. x_2).
2. Make a scatterplot by drawing 100 samples from $\mathcal{N}(\mu + (1, -1)^\top, 2\Sigma)$.
3. Make a scatterplot by drawing 100 samples from a mixture distribution $0.3 \cdot \mathcal{N}\left((1, 0)^\top, \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}\right) + 0.7 \cdot \mathcal{N}\left((-1, 0)^\top, \begin{pmatrix} 1 & -0.2 \\ -0.2 & 1 \end{pmatrix}\right)$.