CSC 480/580 Principles of Machine Learning

04 Linear Classification; Perceptron

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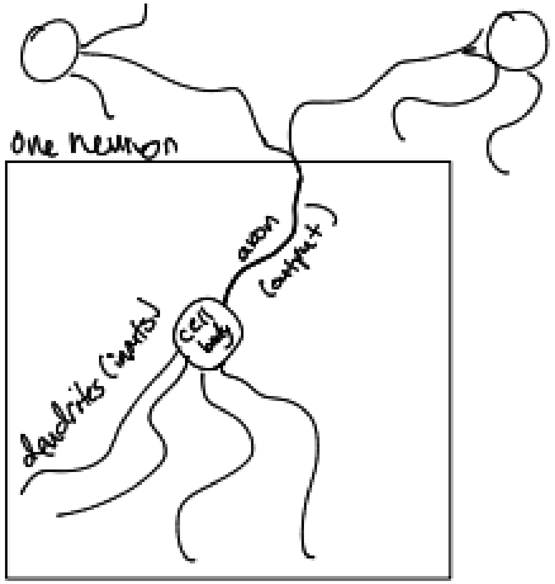
Linear classifiers

- Example application: spam filtering using *bag-of-words* feature representation

	free	offer	lecture	cs	Spam?
Email 1	1	1	0	0	+1
Email 2	0	0	1	1	-1

- If $0.124 \cdot x_{\text{free}} + 2.5 \cdot x_{\text{offer}} + \cdots - 2.31 \cdot x_{\text{lecture}} > 2.12$ then
  - return “spam”
- else
  - return “nonspam”

Linear models: biological motivation

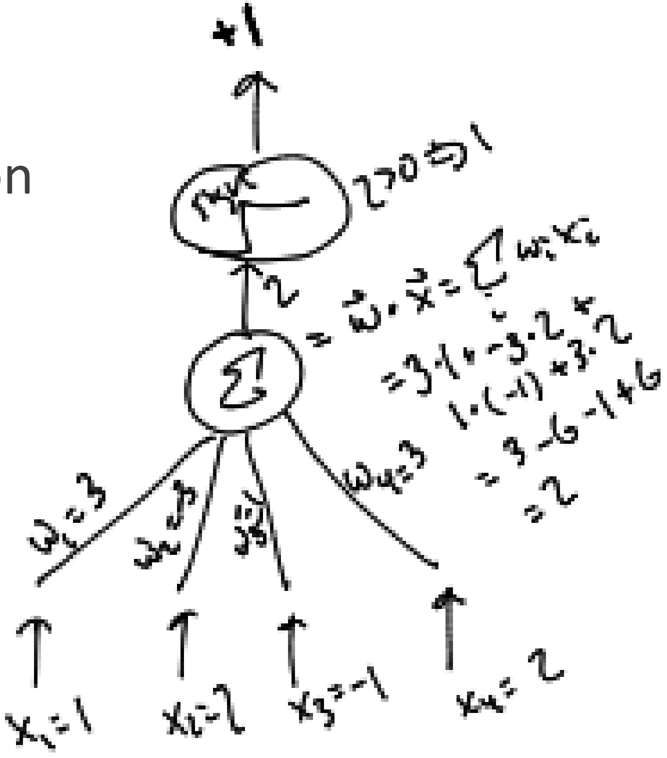


- Firing of a neuron depends on:
  - Whether the incoming neurons are firing
  - The strength of the connections

- The McCulloch-Pitts neural model:

  a neuron Implements a linear threshold function

  $h_w(x) = \text{sign}(\langle w, x \rangle)$



Math review: inner product between vectors

- Given vectors $u, v \in \mathrm{R}^d$, their inner product:

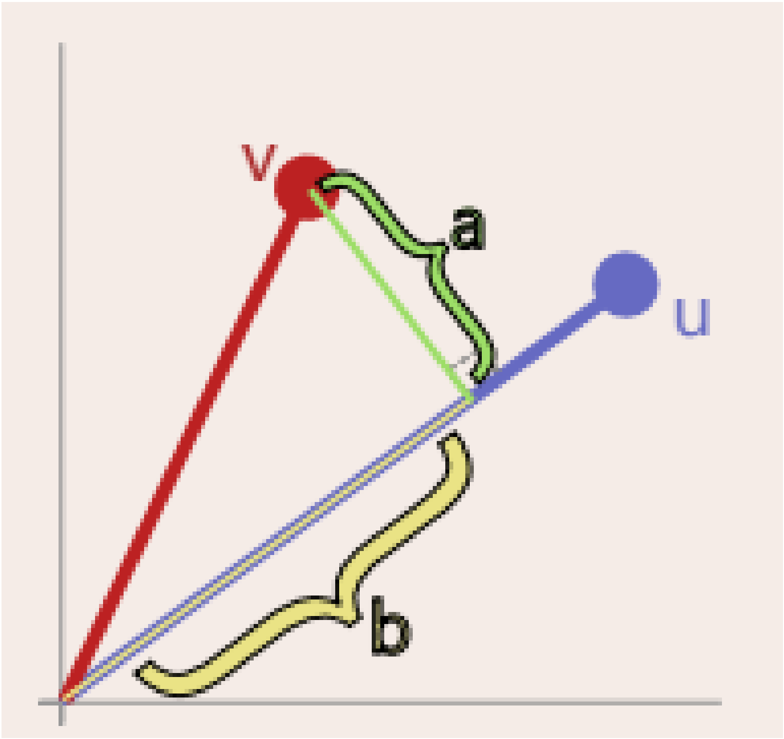
$$\langle u, v \rangle = \sum_{i=1}^{d} u_i \cdot v_i$$

- Geometric interpretation:

$$\langle u, v \rangle = ||u||_2 \cdot ||v||_2 \cdot \cos(\theta(u, v))$$

where $\theta(u, v) \in [0, \pi]$ is the angle between them

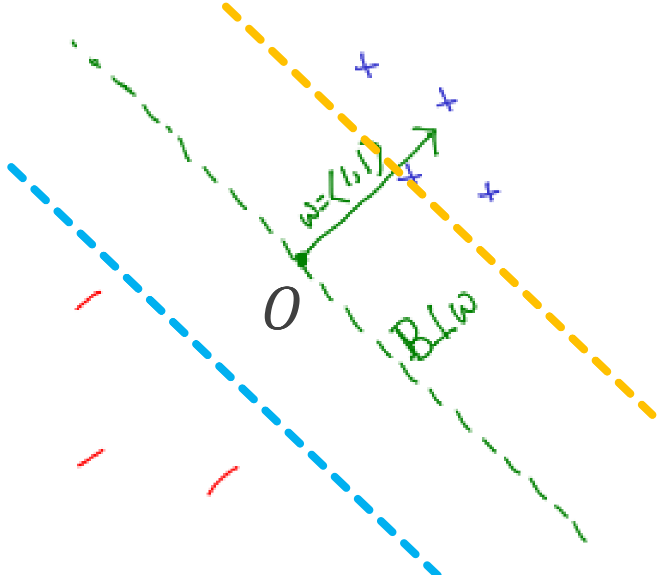
$||v||_2 \cdot \cos(\theta(u, v))$ = (signed) length of v's projection onto u



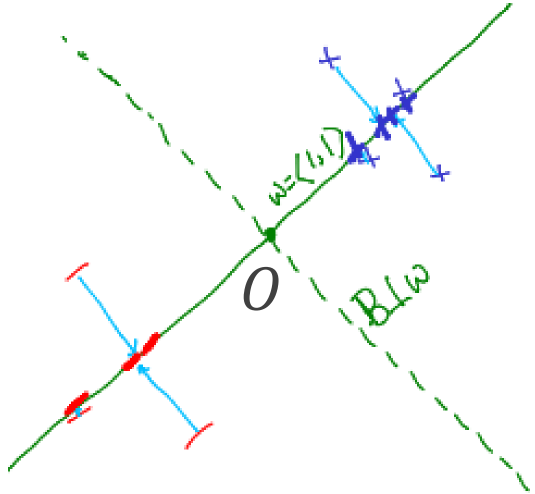
- Observe that $\cos\big(\theta(u, v)\big) \in [-1, +1]$

$\Rightarrow$ Cauchy-Schwarz inequality: $|\langle u, v \rangle| \leq ||u||_2 ||v||_2$

Linear classifiers: geometric view



- Homogeneous linear classifier $h_w(x) = \text{sign}(\langle w, x \rangle)$
- Prediction: Projection + threshold
- Decision boundary: line in 2d, plane in 3d, hyperplane in general



- Non-homogeneous linear classifier $h_{w,b}(x) = \text{sign}(\langle w, x \rangle + b)$

which decision boundary corresponds to offset $b > 0$? Blue or yellow?

- Sometimes convenient to view non-homogeneous. as homogeneous via feature augmentation

$$h_{w,b}(x) = \text{sign}(\langle (w, b), (x, 1) \rangle)$$

$$\downarrow \qquad \downarrow$$

$$\tilde{w} \qquad \tilde{x}$$

Training linear classifiers: The Perceptron algorithm (Rosenblatt, 1958)

- For training *homogeneous* linear classifiers

Initialize $w_1 \leftarrow (0, \dots, 0)$

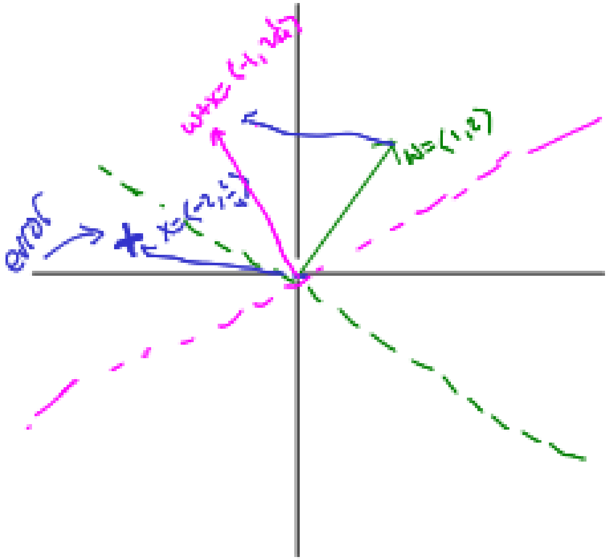
For $t = 1,2, \dots, n$:

Process example $x_t \in \mathrm{R}^d$

Calculate classification score $a_t = w_t \cdot x_t$

Update: if $y_t a_t > 0$: $w_{t+1} \leftarrow w_t$;
otherwise: $w_{t+1} \leftarrow w_t + y_t x_t$.

	free	offer	lecture	cs	Spam?
Email 1	1	1	0	0	+1
Email 2	0	0	1	1	-1



- Properties: (1) Online (2) Error-driven updates

Perceptron for nonhomogeneous linear classifiers

- Idea: reduce to training homogeneous linear classifiers
- $h_{w,b}(x) = \text{sign}(\langle (w,b),(x,1)\rangle) = \text{sign}(\langle \tilde{w},\tilde{x}\rangle)$
- Multiple passes over the data

Algorithm 5 $\textsc{PerceptronTrain}(\mathbf{D}, \mathit{MaxIter})$

$w_d \leftarrow 0$, for all $d = 1 \dots D$ // initialize weights
$b \leftarrow 0$ // initialize bias
for $\mathit{iter} = 1 \dots \mathit{MaxIter}$ **do**
  for all $(x,y) \in \mathbf{D}$ **do**
    $a \leftarrow \sum_{d=1}^{D} w_d\, x_d + b$ // compute activation for this example
    if $ya \le 0$ **then**
      $w_d \leftarrow w_d + y x_d$, for all $d = 1 \dots D$ // update weights
      $b \leftarrow b + y$ // update bias
    end if
  end for
end for
return $w_0, w_1, \dots, w_D, b$

passes

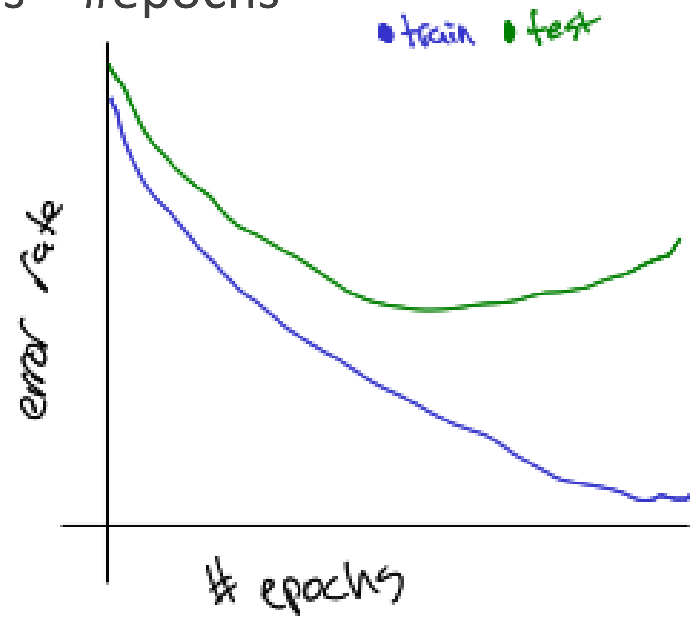
activation = classification score

Algorithm 6 $\textsc{PerceptronTest}(w_0, w_1, \dots, w_D, b, \hat{x})$

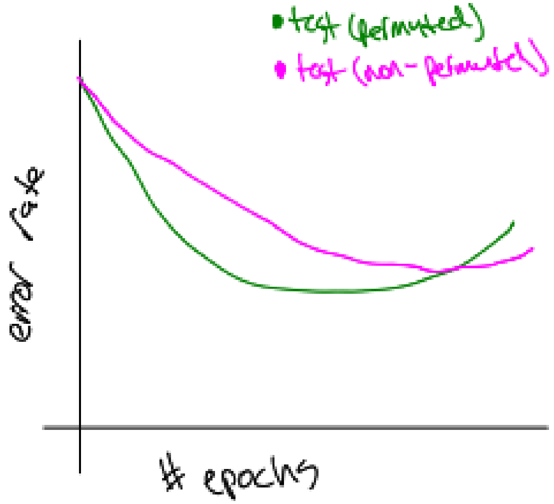
$a \leftarrow \sum_{d=1}^{D} w_d\, \hat{x}_d + b$ // compute activation for the test example
return $\textsc{sign}(a)$

Perceptron: practical issues

- Hyperparameter: MaxIter = #passes = #epochs



- Data shuffling:
  - A non-random training data sequence +++ …. ++ --- …. ---
  - Drawback: only update using the first few examples in each segment
  - More efficient: permute the data sequence for every pass



Perceptron: convergence properties

- Does the Perceptron's iterate w converge?

- Important concept: linear separability
- A dataset S is linearly separable if there exists

 w such that for all $(x, y) \in S$, $y \langle w, x \rangle > 0$

For iter = 1,2,....
    For $(x, y) \in S$:
        Calculate prediction $\hat{y} = \text{sign}(w \cdot x)$
        if $\hat{y} \neq y$, $w \leftarrow w + y\, x$.

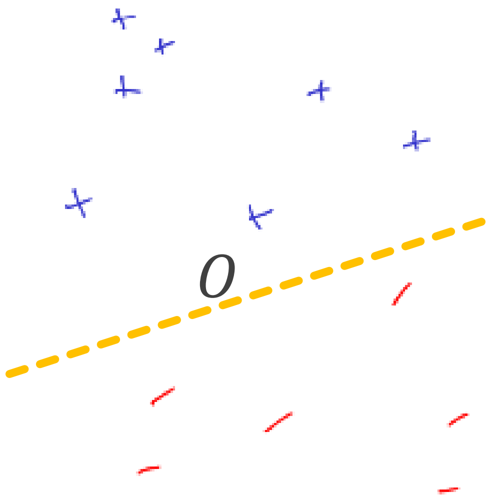


Figure 4.10: separable data

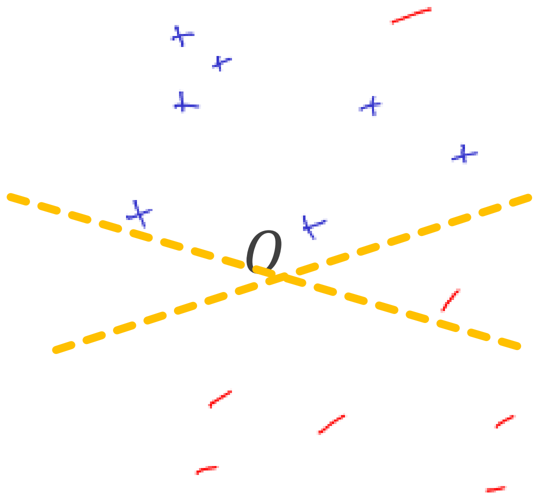


Figure 4.11: inseparable data

Observations:

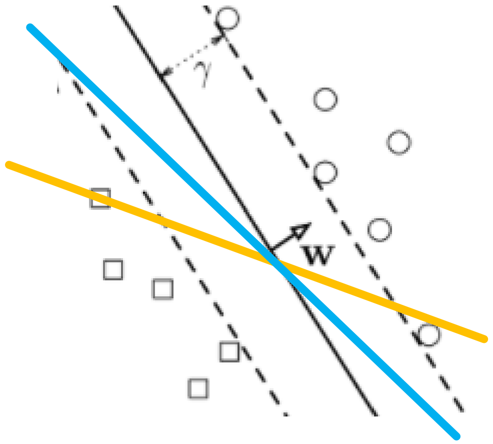
- nonseparable $\Rightarrow$ does not converge
- Separable $\Rightarrow$ converge?
- If so, how long does it take to converge?

Linear classification margins

- Measures easiness of a dataset for linear classification
- Larger margin ⇒ easier dataset ⇒ faster convergence

- Margin of a linear classifier w on S:

$$\text{margin}(S, w) = \begin{cases} \min_{(x,y)\in S} y\langle w, x\rangle, & w \text{ separates } S \\ -\infty, & \text{otherwise} \end{cases}$$



- "Wiggle room" of w on S

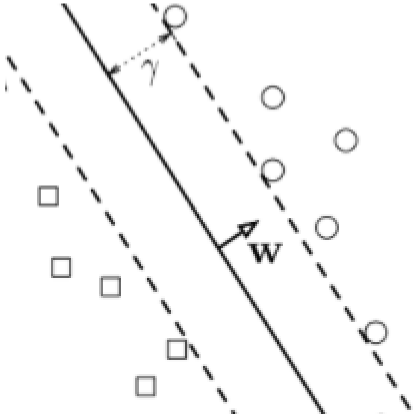
- Margin of dataset S: $\text{margin}(S) = \max_{w:||w||_2=1} \text{margin}(S, w)$
- See book for definition of margins for nonhomogeneous linear classifiers

The Perceptron convergence theorem

Theorem (Perceptron Convergence Theorem, Novikoff 1962): Suppose the Perceptron algorithm is run on a dataset S; Assume:

- $\mathrm{margin}(S) \geq \gamma$, i.e. there exists w^*, $\|w^*\|_2 = 1$, $y\langle w^*, x\rangle \geq \gamma$ for all $(x, y) \in S$
- For all $(x, y) \in S$, $\|x\|_2 \leq 1$

then the Perceptron algorithm makes at most $1/\gamma^2$ updates throughout the process.



- Can also be phrased as an *online learning* mistake bound guarantee

Proof of Perceptron Convergence Theorem

- Denote $w^{(k)}$ the value of w after the k-th update; $w^{(0)} = (0, \dots, 0)$
- Idea: track the progression of $\langle w^{(k)}, w^* \rangle$ and $\left\| w^{(k)} \right\|_2$

- At the k-th update:

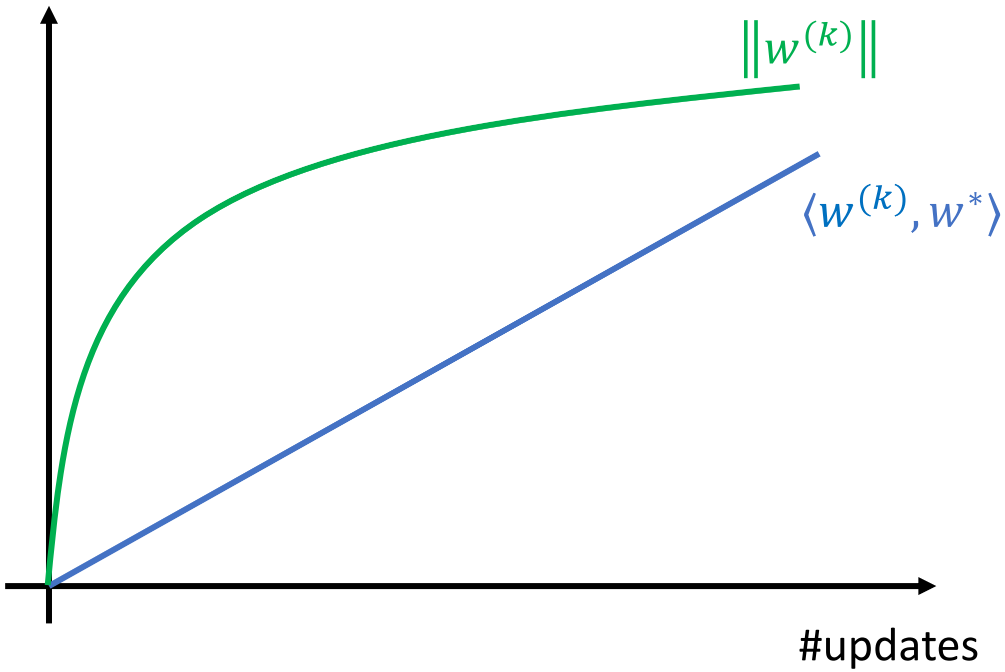
$$\langle w^{(k)}, w^* \rangle = \langle w^{(k-1)} + yx, w^* \rangle \geq \langle w^{(k-1)}, w^* \rangle + \gamma$$

$$\begin{aligned} \left\| w^{(k)} \right\|_2^2 &= \left\| w^{(k-1)} + yx \right\|_2^2 \\ &= \left\| w^{(k-1)} \right\|_2^2 + 2\langle w^{(k-1)}, yx \rangle + \|x\|_2^2 \\ &\leq \left\| w^{(k-1)} \right\|_2^2 + 1 \end{aligned}$$

Proof of Perceptron Convergence Theorem

- Therefore, if a total of k mistakes are made, then:

$$\langle w^{(k)}, w^* \rangle \geq k\,\gamma, \text{ and } \|w^{(k)}\| \leq \sqrt{k}$$



Proof of Perceptron Convergence Theorem

- Let M = #mistakes made up to time step n

$$\langle w_{n+1}, w^* \rangle \geq M\,\gamma, \text{ and } \|w_{n+1}\| \leq \sqrt{M}$$

- Meanwhile, by Cauchy-Schwarz,

$$\langle w_{n+1}, w^* \rangle \leq \|w_{n+1}\| \cdot \|w^*\| = \|w_{n+1}\|$$

- This implies that $M\,\gamma \leq \sqrt{M} \Rightarrow M \leq 1/\gamma^2$
- This holds for all n, which concludes the proof

Practical versions: voting Perceptron

- Naïve Perceptron: return the last iterate $w^{(K)}$
- Drawback:
  - say making one pass, last example is an outlier
  - Last update may ruin a previously trained good model

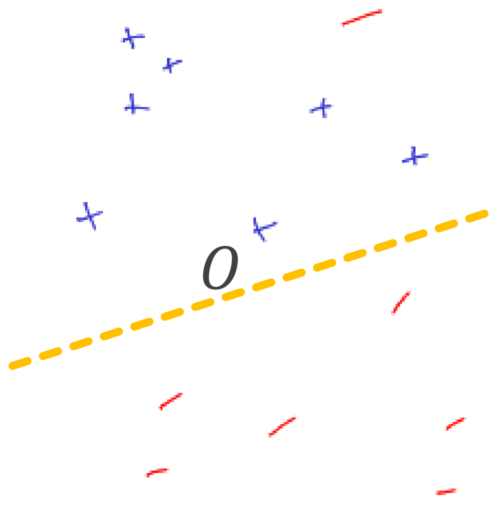


Figure 4.11: inseparable data

- A more robust output classifier:

$$h(x) = \text{sign}\left(\sum_{t=1}^{T} h_t(x)\right) = \text{sign}\left(\sum_{k=0}^{K} c^{(k)} \boxed{h_{w^{(k)}}(x)}\right)$$

$\boxed{h_{w^{(k)}}(x)} \in \{-1, +1\}$

$h_t(x)$: Linear classifier at iteration t

$c^{(k)}$: Number of times t when $h_t = h_{w^{(k)}}$

- Has good predictive performance, but computationally expensive to maintain

Practical versions: averaged Perceptron

- $h(x) = \operatorname{sign}(\langle \bar{w}, x \rangle)$, where $\bar{w} = \frac{1}{T}\sum_t w_t = \frac{1}{\sum_{k=0}^{K} c^{(k)}} \sum_{k=0}^{K} c^{(k)}\, w^{(k)}$ is the averaged predictor

- This is equivalent to $\operatorname{sign}\left(\left\langle \sum_{k=0}^{K} c^{(k)}\, w^{(k)}, x \right\rangle\right)$

- Efficient implementation

  (avoid extensive bookkeeping when no update)

- Exercise: show that the final output is $\bar{w}$

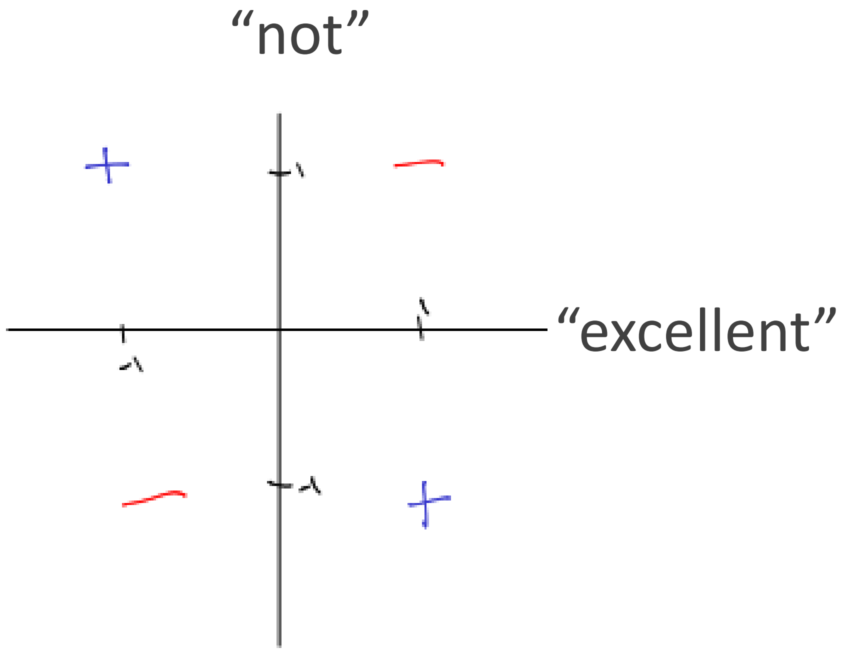
Algorithm 7 AveragedPerceptronTrain(**D**, *MaxIter*)

```
w ← ⟨0, 0, ... 0⟩ , b ← 0                    // initialize weights and bias
u ← ⟨0, 0, ... 0⟩ , β ← 0                    // initialize cached weights and bias
c ← 1                                        // initialize example counter to one
for iter = 1 ... MaxIter do
  for all (x,y) ∈ D do
    if y(w · x + b) ≤ 0 then
      w ← w + [y x]   → z^(k)                // update weights
      b ← b + y                              // update bias
      u ← u + y c x                          // update cached weights
      β ← β + y c                            // update cached bias
    end if
    c ← c + 1                                // increment counter regardless of update
  end for
end for
return w - (1/c) u, b - (1/c) β              // return averaged weights and bias
```

$$\sum_{k=0}^{K} z^{(k)} \qquad \sum_{k=0}^{K} c^{(k)} \qquad \sum_{k=0}^{K} z^{(k)} \sum_{l<k} c^{(l)}$$

Perceptron: limitations

- The ‘XOR’ problem: data linearly nonseparable



- E.g. sentiment analysis

- Possible fix: introduce nonlinear feature maps

$$x = (x_1, x_2) \mapsto \phi(x) = (x_1,\ x_2,\ x_1 x_2,\ x_1^2,\ x_2^2)$$, e.g. containing “mega-feature” $x_{\text{no}} \cdot x_{\text{excellent}}$

- Later in the course: kernel methods (high/infinite dim ϕ); neural networks (learn ϕ from data)

Next lecture (2/8)

- Practical issues: feature selection; feature transformation; model performance evaluation

- Assigned reading: CIML Sections 5.1-5.6